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An Interior-Point Path-Following Method for Computing a Perfect Stationary Point of a Polynomial Mapping on a Polytope*

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Abstract

As a refinement of the concept of stationary point, the notion of perfect stationary point was formulated in the literature. Although simplicial methods could be applied to approximate such a point, these methods do not make use of the possible differentiability of the problem and can be very time-consuming even for small-scale problems. To fully exploit the differentiability of the problem, this paper develops an interior-point path-following method for computing a perfect stationary point of a polynomial mapping on a polytope. By incorporating a logarithmic barrier term into the linear objective function

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with an appropriate convex combination, the method closely approximates some stationary points of the mapping on a perturbed polytope, especially when the perturbation is sufficiently small. It is proved that there exists a smooth path which starts from a point in the interior of a polytope and ends at a perfect stationary point. A predictor-corrector method is adopted for numerically following the path. Numerical results further confirm the effectiveness of the method.

Keywords: Variational Inequality Problem, Perfect Stationary Point, Interior-point Path-following Method, Predictor-Corrector Method

1 Introduction

Let $P = \{x \in R^n \mid Ax \leq b\}$, where A is an $m \times n$ matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and $b = (b_1, b_2, \ldots, b_m)^\top$. Let $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^\top$ be a (point-to-point) mapping from P to R^n. The stationary point or variational inequality problem of f on P with respect to f can be stated as follows. Find a point $x^* \in P$ such that, for all $x \in P$,

$$(x - x^*)^\top f(x^*) \leq 0.$$

Many important problems in fields such as economics, engineering, finance, game theory, and mathematical optimizations can be formulated as a stationary point problem, which are referred to Facchinei and Pang (2003) and the references therein. To study their properties, a great effort has been made in the literature. A comprehensive and unified mathematical treatment on the existence, convergence and sensitivity of stationary points can be found in Rockaffelar and Wets (2009) and the references therein. It is documented that there always exists a stationary point of f on P if f is a continuous mapping and P is, more generally, a nonempty convex and compact set (Hartman and Stampacchia, 1966 and Eaves, 1971). Nevertheless, as pointed out in Myerson (1978) and Selten (1975) for strategic games, there can have multiple stationary points and some of these may be inconsistent with intuitive notions about what could be stationary. To reduce this ambiguity and eliminate some of these counterintuitive stationary points, as a straightforward extension of a perfect equilibrium for strategic games in Selten (1975), the notion of a perfect stationary point of f on P is introduced in van der Laan et al. (2006).

The concept of a perfect stationary point is a strict refinement of a stationary point. In case of continuity of f and more general nonemptiness, convexity and compactness of P, one can show that there always exists a perfect stationary point of f on P. With this existence of a perfect stationary point of f on P, how to effectively and efficiently compute such a point is an important and challenging issue in its applications. Govindan and Klumpp (2002) find that a direct check for whether a stationary point is perfect requires, in principle, an infinite number of computations, and Hansen et al. (2010) prove that it is NP-hard to determine whether a given stationary point is perfect. Following the definition, one can find a perfect stationary point on a polytope by computing stationary points for a sequence of perturbed polytopes and obtaining a perfect stationary point as their limit. The efficiency of such an approach depends on the sequence and its underlying method for computing stationary points of f on the perturbed polytopes and can be very time-consuming.

The computation of stationary points for a continuous mapping on a convex compact set has been of great interest in the academic and professional communities ever since the seminal work by Lemke and Howson (1964) and the epoch-making breakthrough by Scarf (1967). A survey and exposition of some representatives of algorithms for computing a stationary point can be found in Allgower and Georg (2003), Argyros and Hilout (2009), Facchinei and Pang (2003), Harker and Pang (1990), and the references therein. Among these algorithms, the path-following approach which can be categorized into a piecewise linear type and a smooth type, is considered as one of the most successful paradigms for computing stationary points. Simplicial methods, a piecewise linear type of path-following methods, are a powerful mechanism on computing stationary points for highly nonlinear or nonsmooth mappings, but they do not make use of the possible differentiability of the problem and can be very time-consuming even when the dimension of the problem is low. To overcome this deficiency and exploit differentiability of a problem in the computation of stationary points, smooth path-following methods are developed in the literature, which are referred to Borkovsky et al. (2010), Chen and Harker (1995), Chen and Ye (1999), Chow et al. (1978), Fan and Yu (2009), Zhou and Yu (2014), etc. Smooth path-following methods for computing stationary points on some specific polytopes are proposed by Harsanyi and Selten (1988), Herings and Peeters (2001, 2010) and Govindan and Wilson (2003, 2010).

Few methods have been designed specifically for computing a perfect stationary point. Van der Laan et al. (1998, 2006) and Yang (1996) study the stability of stationary point of continuous function on polytopes. They generalize the concepts of perfect and proper Nash equilibrium into stationary points and define the more refined notion of robust stationary point. They prove that a robust stationary point exists for continuous mappings on polytopes and develope a simplicial algorithm to approximate such a point. As the differentiability of the problem is not taken into account in a simplicial algorithm, a question naturally is raised that whether a smooth path-following method can be developed to compute a perfect stationary point of a polynomial mapping on a polytope.

This paper develops an effective interior-point path-following method for computing a perfect stationary point of a polynomial mapping on a polytope. The basic idea of the method is to closely approximate stationary points of the mapping on a perturbed polytope derived from the original polytope. By introducing an extra variable varying between zero and one, we obtain a barrier objective function from an appropriate convex combination of a logarithmic barrier term and the linear objective function with the mapping of the point as its coefficients. An application of the barrier objective function leads to a barrier variational inequality problem that deforms from a trivial problem to the original one. The logarithmic barrier term forces the stationary points of the barrier problem to stay in the interior of the polytope. When the artificial variable is sufficiently small, the corresponding coefficient of the barrier term ensures that every barrier stationary point to the variational inequality problem is an ε-perfect stationary point of the original problem for an arbitrarily small $\varepsilon > 0$. It is proven that there exists a smooth path leading to a perfect stationary point. A system of differential equations for following the smooth path is formulated by parameterizing each point on the path with the corresponding arc length. A predictor-corrector method is adopted for solving the system of differential equations. Numerical experiments further show that the method is effective.

The rest of this paper is organized as follows. A smooth path to a perfect stationary point of a polynomial mapping on polytopes is constructed in

Section 2. Numerical performance of the method is presented in Section 3. The paper is concluded in Section 4.

2 Construction of a Smooth Path to a Perfect Stationary point of a Polynomial Mapping on a Polytope

2.1 Perfect Stationary point

Let $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$. For $i \in M$, let $a_i^\top = (a_{i1}, a_{i2}, \ldots, a_{in})$ be the i-th row of some given $m \times n$-matrix A. We assume that $P = \{x \in R^n | Ax \leq b\}$ is a polytope and has a nonempty interior. The interior of P is denoted by $\mathrm{int}(P)$. Let $f : P \to R^m$ be a continuous (point-to-point) mapping. Since P is compact, for every $x \in P$, there exists a vector $z = (z_1, z_2, \ldots, z_m)^\top \geq 0$ such that $f(x) = \sum_{i=1}^m a_i z_i$.

Definition 1. For given $\varepsilon > 0$, $x \in \mathrm{int}(P)$ is an ε-perfect stationary point of f if there exists a vector $z = (z_1, z_2, \ldots, z_m)^\top \geq 0$ satisfying that $f(x) = A^\top z$ and $a_i^\top x \geq b_i - \varepsilon$ for any $i \in M$ with $z_i > 0$.

$x^* \in P$ is a perfect stationary point of f if x^* is the limit of a convergent sequence of points $x(\varepsilon_k)$, $k = 1, 2, \ldots$, with $\lim_{k\to\infty} \varepsilon_k = 0$, where $x(\varepsilon_k)$ is an ε_k-perfect stationary point of f for all k.

Example 1. Let $a_1 = -a_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a_2 = -a_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $b_1 = b_2 = 1$, and $b_3 = b_4 = 0$. Then $P = \{x \in R^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ is the unit square. Let $f(x) = \begin{pmatrix} x_1 \\ -1 \end{pmatrix}$. Clearly, x^* is a stationary point of f if and only if it holds that

$$
\begin{aligned}
f_i(x^*) &\leq 0 \text{ if } x_i^* = 0,\\
f_i(x^*) &= 0 \text{ if } 0 < x_i^* < 1,\\
f_i(x^*) &\geq 0 \text{ if } x_i^* = 1.
\end{aligned}
$$

Thus, the set of stationary points of f is equal to $\{(0,0)^\top, (1,0)^\top\}$. However, only $(1,0)^\top$ is a perfect stationary point of f on P. To show the latter, for $\varepsilon \in (0,1)$, let

$$x_1(\varepsilon) = (1-\varepsilon)^{1/2},\ x_2(\varepsilon) = \varepsilon.$$

Clearly, $x(\varepsilon)$ lies in the interior of P for each $\varepsilon \in (0,1)$. Note that $f(x(\varepsilon)) = z_1a_1 + z_4a_4 = z_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_4\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, with $z_1 = (1-\varepsilon)^{1/2} > 0$ and $z_4 = 1 > 0$. It also holds that $a_1^\top x(\varepsilon) = (1,0)\begin{pmatrix} (1-\varepsilon)^{1/2} \\ \varepsilon \end{pmatrix} = (1-\varepsilon)^{1/2} \geq 1-\varepsilon = b_1 - \varepsilon$, and $a_4^\top x(\varepsilon) = (0,-1)\begin{pmatrix} (1-\varepsilon)^{1/2} \\ \varepsilon \end{pmatrix} = -\varepsilon \geq b_4 - \varepsilon$. Hence, for every $\varepsilon \in (0,1)$, $x(\varepsilon)$ is an ε-perfect stationary point of f. It is easy to see that $\lim_{\varepsilon \to 0} x(\varepsilon) = (1,0)$. The point $(0,0)^\top$ is not a perfect stationary point of f, because any ε-perfect stationary point of f implies $f_1(x(\varepsilon)) > 0$ and therefore $x_1(\varepsilon) \geq 1-\varepsilon$, so that $x_1(\varepsilon)$ cannot converge to zero when ε goes to zero.

The example shows that the concept of perfect stationary point is a strict refinement of stationary point.

2.2 Construction of a Smooth Path

From now on we assume that f is a polynomial mapping with degree of at most $q-1$. For any $t \in (0,1]$, let $P(t) = \{x \in R^n \mid a_i^\top x + t\eta \leq b_i,\ i = 1,2,\ldots,m\}$, where η is any given positive number such that $P(1)$ has an interior point. Then a stationary point of f on the perturbed polytope $P(t)$ is a solution to the following variational inequality problem with respect to f and $P(t)$: Find a point $x^*(t) \in P(t)$ satisfying that

$$(x - x^*(t))^\top f(x^*(t)) \leq 0$$

for all $x \in P(t)$. As a result of the continuity of f and the nonemptiness and boundedness of $P(t)$, there always exists a stationary point of f on $P(t)$. From the theory of linear programming, it is asserted that, for every stationary point $x^*(t)$ of f on $P(t)$, there exists a vector $z^*(t) \geq 0$ in R^m satisfying that $f(x^*(t)) = A^\top z^*(t)$ and $a_i^\top x^*(t) = b_i - t\eta$ for all $i \in M$ with $z_i^*(t) > 0$. This assertion implies that every stationary point of f on $P(t)$ is an ε-perfect stationary point of f on P, where $\varepsilon = t\eta$.

In order to construct a smooth path that leads to a perfect stationary point of f on P, we will closely approximate some stationary points of f on $P(t)$ especially when $t \in (0, 1]$ is sufficiently small. For $t \in (0, 1]$ and any given x in the interior of $P(t)$, consider the unconstrained optimization problem

$$\max_y \; (1 - t^q) y^\top f(x) + t^q \sum_{i=1}^m \ln(b_i - a_i^\top y - t\eta). \tag{1}$$

The first order optimality condition of (1) together with $y = x$ leads to the system

$$\begin{aligned} &(1 - t^q) f(x) - \textstyle\sum_{i=1}^m z_i a_i = 0, \\ &a_j^\top x + s_j + t\eta - b_j = 0, \; j = 1, 2, \ldots, m, \\ &z_j s_j = t^q, \; j = 1, 2, \ldots, m. \end{aligned} \tag{2}$$

Let α be a vector of R^n with sufficiently small length $\|\alpha\|$. Subtracting a perturbation term of $t^q(1-t^q)\alpha$, we obtain the system

$$\begin{aligned}
&(1-t^q)f(x) - \textstyle\sum_{i=1}^m z_i a_i - t^q(1-t^q)\alpha = 0, \\
&a_j^\top x + s_j + t\eta - b_j = 0,\ j = 1, 2, \ldots, m, \\
&z_j s_j = t^q,\ j = 1, 2, \ldots, m.
\end{aligned} \tag{3}$$

When $t = 0$, one can see that the system (3) becomes the stationary point condition for the original variational inequality problem. Let $p(x, z, s, t; \alpha)$ denote the left side of system (3). For any $\alpha \in R^n$, let $p_\alpha(x, z, s, t) = p(x, z, s, t; \alpha)$.

Lemma 1. *For any given $\alpha \in R^n$, when $t = 1$, $p_\alpha(x, z, s, t) = 0$ has a unique solution $(x^*(1), z^*(1), s^*(1), 1)$.*

Proof. When $t = 1$, the system $p_\alpha(x, z, s, t) = 0$ becomes

$$\begin{aligned}
&-\textstyle\sum_{i=1}^m z_i a_i = 0, \\
&a_j^\top x + s_j + \eta - b_j = 0,\ j = 1, 2, \ldots, m, \\
&z_j s_j = 1,\ j = 1, 2, \ldots, m.
\end{aligned} \tag{4}$$

This system comes from the first order optimality condition of the problem

$$\max_y \sum_{i=1}^m \ln(b_i - a_i^\top y - \eta). \tag{5}$$

Direct calculation shows that the Hessian matrix of the objective function is $-A^\top CA$, where C is a diagonal matrix with its i-th diagonal element given by $1/(b_i - a_i^\top y - \eta)^2$. Since A is of full column rank, it follows that $-A^\top CA$ is negative definite, and hence that the objective function is strictly concave. So the solution of optimization problem (5) must be unique. □

Let $\eta = \frac{\eta_0}{2}$, where η_0 is the solution of the linear program

$$\begin{aligned}&\max \eta\\ &\text{s.t. } a_i^\top x + \eta \leq b_i, i = 1, \ldots, m.\end{aligned}$$

When $t = 1$, $x^*(1)$ is the solution of the convex program

$$\max_y \sum_{i=1}^m \ln(b_i - a_i^\top y - \eta).$$

Then, we obtain $s^*(1)$ and $z^*(1)$ from

$$\begin{aligned} s^*(1) &= b - \eta e - Ax^*(1),\\ z_j^*(1) &= \frac{1}{s_j^*(1)},\ j = 1, 2, \ldots, m, \end{aligned}$$

where $e = (1, 1, ..., 1) \in R^m$.

For any given $(x, z, s, t) \in P \times R_+^m \times R_+^m \times [0, 1]$, we define $\varphi(x, z, s, t)$ to be the set of all $(\hat{x}, \hat{z}, \hat{s})$ satisfying the system

$$\begin{aligned} &(1 - t^q)f(x) - \textstyle\sum_{i=1}^m \hat{z}_i a_i - t^q(1 - t^q)\alpha = 0,\\ &a_j^\top \hat{x} + \hat{s}_j + t\eta - b_j = 0,\ j = 1, 2, \ldots, m, \\ &\hat{z}_j \hat{s}_j = t^q,\ j = 1, 2, \ldots, m. \end{aligned} \tag{6}$$

Lemma 2. *For any $(x, z, s, t) \in P \times R_+^m \times R_+^m \times [0, 1]$, $\varphi(x, z, s, t)$ is a nonempty, convex and compact subset of $P \times R_+^m \times R_+^m$.*

Proof. For any given $(x, z, s, t) \in P \times R_+^m \times R_+^m \times (0, 1]$, consider convex programs

$$\max_y\ (1 - t^q)y^\top f(x) + t^q \textstyle\sum_{i=1}^m \ln(b_i - a_i^\top y - t\eta) - t^q(1 - t^q)\alpha y, \tag{7}$$

and for any given $(x, z, s, t) \in P \times R_+^m \times R_+^m \times \{0\}$, consider linear programs

$$\begin{aligned} &\max_y\ y^\top f(x)\\ &s.t.\ a_i^\top y \leq b_i, i \in M. \end{aligned} \tag{8}$$

An application of the first order optimality condition to these convex and linear programs together with $y = \hat{x}$ is given by the system

$$\begin{aligned}&(1-t^q)f(x) - \textstyle\sum_{i=1}^m \hat{z}_i a_i - t^q(1-t^q)\alpha = 0,\\ &a_j^\top \hat{x} + \hat{s}_j + t\eta - b_j = 0,\ j = 1, 2, \ldots, m, \\ &\hat{z}_j \hat{s}_j = t^q,\ j = 1, 2, \ldots, m,\end{aligned} \tag{9}$$

which is the same as the system (6). Thus $\varphi(x, z, s, t)$ is a nonempty, convex and closed set. Let $(\hat{x}, \hat{z}, \hat{s})$ be a solution of the system (6). Then from the system (6), we get:

$\hat{s}_j = b_j - t\eta - a_j^\top \hat{x}$ is bounded due to the boundedness of P.

Consider $t = 0$. $\hat{z}_j = 0$ when $\hat{s}_j > 0$ with $a_j^\top \hat{x} - b_j < 0$ and $\hat{z}_j > 0$ when $\hat{s}_j = 0$ with $a_j^\top \hat{x} - b_j = 0$. Let $L(\hat{x}) = \{i \mid a_j^\top \hat{x} - b_j = 0\}$, then $f(x) = \sum_{i \in L} \hat{z}_i a_i$. For any interior point x^0, $(x^0 - \hat{x})^\top f(x) = \sum_{i \in L} z_i a_i^\top (x^0 - \hat{x})$ with $a_i^\top (x^0 - \hat{x}) = a_i^\top x^0 - b_i < 0$. Since $f(x)$ and $x^0 - \hat{x}$ are bounded, $\hat{z}_i$ is bounded.

Consider $t > 0$. For any given t, $\hat{s}_j = b_j - t\eta - a_j^\top \hat{x} > 0$ and $\hat{z}_j = \frac{t^q}{\hat{s}_j}$. Because of boundedness and positivity of $\hat{s}_j$, $\hat{z}_j$ is bounded.

The lemma follows immediately. □

Let $\Lambda = \{s \in R_+^m \mid s_j \leq \gamma_0 \text{ for all } j \in M\}$ and $\Omega = \{z \in R_+^m \mid z_j \leq \gamma_1 \text{ for all } j \in M\}$, where γ_0 and γ_1 are upper bounds for s and z, respectively. As a result of Lemma 2, it follows that $\varphi : P \times R_+^m \times R_+^m \times [0,1] \mapsto P \times \Lambda \times \Omega$ is an upper semi-continuous (point-to-set) mapping. Since $P \times \Lambda \times \Omega$ is a nonempty compact and convex set, we obtain from Kakutani's fixed point theorem that, for any $t \in [0,1]$, $\varphi(\cdot, t) : P \times \Lambda \times \Omega \mapsto P \times \Lambda \times \Omega$ has a fixed point in $P \times \Lambda \times \Omega$.

To continue the development, we need the following fixed point theorem from Mas-Colell (1974).

Theorem 1. *Let Ψ be a nonempty, compact and convex subset of R^r and $h : \Psi \times [0,1] \mapsto \Psi$ an upper semi-continuous mapping. Then the set $H = \{(z,t) \in \Psi \times [0,1] \mid z \in h(z,t)\}$ contains a connected set H^c such that $(\Psi \times \{1\}) \cap H^c \neq \emptyset$ and $(\Psi \times \{0\}) \cap H^c \neq \emptyset$.*

Let Δ denote the set of all $(x^*(t), z^*(t), s^*(t), t) \in P \times R^m_+ \times R^m_+ \times [0,1]$ satisfying the system (3). Comparing the systems (3) and (6), one can see that $(x^*(t), z^*(t), s^*(t)) \in \varphi(x^*(t), z^*(t), s^*(t), t)$ if and only if $(x^*(t), z^*(t), s^*(t), t)$ is a solution of the system (3). Thus, $\Delta = \{(x^*(t), z^*(t), s^*(t), t) \in P \times \Lambda \times \Omega \times [0,1] \mid (x^*(t), z^*(t), s^*(t)) \in \varphi(x^*(t), z^*(t), s^*(t), t)\}$. Therefore, as a corollary of Mas-Colell's fixed point theorem, we obtain the following result.

Corollary 1. *Δ has a component that intersects both the sets $P \times R^m_+ \times R^m_+ \times \{1\}$ and $P \times R^m_+ \times R^m_+ \times \{0\}$.*

According to Schanuel et al. (1991), we know that Δ is a semi-algebraic set since it is described by a finite number of polynomials. Thus all the components of Δ are path-connected. Therefore, any two points in a component of Δ can be joined by a path. Lemma 1 implies that there is a unique component of Δ intersecting $P \times R^m_+ \times R^m_+ \times \{1\}$. This result together with Corollary 1 shows that the component must also intersect the set $P \times R^m_+ \times R^m_+ \times \{0\}$.

Next we show that there exists a smooth path in Δ starting from the unique solution $(x^*(1), z^*(1), s^*(1), 1)$ of the system (3) with $t = 1$ and ending at a unique point in the set $P \times R^m_+ \times R^m_+ \times \{0\}$. An application of the transversality theorem in Eaves and Schmedders (1999) leads to the following result.

Theorem 2. *For almost all α, there exists a smooth path in Δ that starts from the unique solution $(x^*(1), z^*(1), s^*(1), 1)$ on the level $t = 1$ and ends at a unique point on the target level $t = 0$.*

Proof. It is proved in the appendix that the Jacobian matrix of $p(x, z, s, t; \alpha)$ is of full-row rank for any $(x, z, s, t; \alpha) \in \text{int}(P(t)) \times R^m_{++} \times R^m_{++} \times (0, 1] \times R^n$. With $t = 1$, it follows immediately that zero is a regular value of $p_\alpha(x, z, s, 1)$ on $\text{int}(P(t)) \times R^m_{++} \times R^m_{++}$. One can see from the system (3) that $p(x, z, s, t; \alpha)$ is a smooth function on the open set $\text{int}(P(t)) \times R^m_{++} \times R^m_{++} \times (0, 1) \times R^n$. These results together with a direct application of the transversality theorem in Eaves and Schmedders (1999) show that zero is a regular value of $p_\alpha(x, z, s, t)$ on $\text{int}(P(t)) \times R^m_{++} \times R^m_{++} \times (0, 1)$ for almost all $\alpha \in R^n$.

Let us choose a vector $\alpha \in R^n$ having sufficiently small $\|\alpha\|$ and satisfying that zero is a regular value of $p_\alpha(x, z, s, t)$ on $\text{int}(P(t)) \times R^m_{++} \times R^m_{++} \times (0, 1]$. An application of the well-known Implicit Function Theorem leads to that there exists a smooth path in Δ starting from the unique solution $(x^*(1), z^*(1), s^*(1), 1)$ of $p_\alpha(x, z, s, t) = 0$. Clearly, the path is not tangential to $\text{int}(P(t)) \times R^m_{++} \times R^m_{++} \times \{1\}$ since the Jacobian matrix of $p_\alpha(x, z, s, t)$ at the starting point is of full-row rank. For any given $t \in (0, 1)$, one can easily derive that $a_j^\top x^*(t) + t\eta < b_j$ since $s_j^*(t)$ is bounded and $z_j^*(t)s_j^*(t) - t^q = 0$. Thus the path cannot hit the boundary of $P(t) \times R^m_+ \times R^m_+$. Therefore we obtain from Corollary 1 and the path-connectedness of each component in Δ that there exists a smooth path in Δ that starts from $(x^*(1), z^*(1), s^*(1), 1)$ and ends at a unique point on the target level $t = 0$. This completes the proof. □

2.3 Existence of a Smooth Path to a Perfect Stationary Point of a Polynomial Mapping on a Polytope

Let $\bar{A}$ be an invertible matrix consisting of n rows of A. We assume without loss of generality that $\bar{A} = (a_1, a_2, ..., a_n)^\top$. Let $\bar{b} = (b_1, b_2, ..., b_n)^\top$ and $\bar{s}(t) = (s_1(t), s_2(t), ..., s_n(t))^\top$. Note that $\bar{b}$ and $\bar{s}(t)$ are the vectors composed of the components of b and $s(t)$ corresponding to those rows of A forming $\bar{A}$, respectively. With these notations, we obtain from the system (3) that

$$x(t) = \bar{A}^{-1}\bar{b} - t\eta\bar{A}^{-1}e - \bar{A}^{-1}\bar{s}(t) = \bar{A}^{-1}\bar{b} - \bar{A}^{-1}(t\eta e + \bar{s}(t)),$$

where $e \in R^n$ is the vector of ones. Thus, for any $i \in N$, $f_i(x(t))$ can be rewritten as a polynomial function of $t\eta + s_l(t)$, $l = 1, 2, ..., n$, with degree of at most $q-1$ and $s_l(t) > 0$. Therefore the absolute value of each term of $f_i(x(t))$ is at least $c_0 t^{q-1}$ for all $i \in N$, where c_0 is some positive constant.

Let $(x(t_k), z(t_k), s(t_k), t_k) \in \Delta$, with $t_k > 0$, $k = 1, 2, ...$, be any given sequence on the path with $\lim\limits_{k\to\infty} t_k = 0$. It follows from Theorem 2 that $(x(t_k), z(t_k), s(t_k))$ is a convergent sequence. Let

$$(x^*, z^*, s^*) = \lim_{k\to\infty} (x(t_k), z(t_k), s(t_k)).$$

We obtain from the system (3) that x^* is a stationary point of the original variational inequality problem. Let

$$\delta(t_k) = \|(x(t_k), z(t_k), s(t_k)) - (x^*, z^*, s^*)\|, k = 1, 2, \ldots.$$

Then, $\lim\limits_{k\to\infty} \delta(t_k) = 0$.

Theorem 3. *When k is sufficiently large, $x(t_k)$ is an ε_k-perfect stationary point of f on P.*

Proof. Let $J(s^*) = \{i \mid s_i^* > 0\}$. Thus, $s_j^* = 0$ for any $j \notin J(s^*)$. Therefore, $s_j(t_k) \leq \delta(t_k)$ for all k and $j \notin J(s^*)$. This result together with the system (3) ensures that, for all k and $j \notin J(s^*)$, $z_j(t_k) \geq t_k^q/\delta(t_k) > O(t_k^q)$ and

$$a_j^\top x(t_k) = b_j - s_j(t_k) - t_k\eta \geq b_j - \delta(t_k) - t_k\eta = b_j - (\delta(t_k) + t_k\eta). \qquad (10)$$

Moreover, for all k and $j \in J(s^*)$, it follows from $z_j(t_k)s_j(t_k) = t_k^q$ and $\lim\limits_{k\to\infty} s_j(t_k) = s_j^* > 0$ that

$$z_j(t_k) = O(t_k^q).$$

Let $\varepsilon_k = \delta(t_k) + t_k\eta$. Then, $\lim_{k\to\infty} \varepsilon_k = 0$. Since the absolute value of each term of $f_i(x(t_k))$ is at least $c_0 t_k^{q-1}$, $z_j(t_k) = O(t_k^q)$ for all $j \in J(s^*)$, and $f_i(x(t_k)) - \sum_{j\in M} a_{ji} z_j(t_k) - t_k^q(1-t_k^q)\alpha_i = 0$ for all $i \in N$, hence, one can treat, in terms of numerical computation, $z_j(t_k)$ as zero for all $j \in J(s^*)$ when k is sufficiently large. Thus, it follows from the inequality (10) that $x(t_k)$ is an ε_k-perfect stationary point of f for all sufficiently large k. This completes the proof. □

As a corollary of Theorem 3, we come to the following conclusion.

Corollary 2. *There exists a smooth path in Δ leads to a perfect stationary point of f on P.*

Proof. It follows from Theorem 2 that there exists a smooth path in Δ leading from $x^*(1)$ to a stationary point x^* of f on P. Let $\{x(t_k) \mid k = 1, 2, ...\}$ be any given sequence on the path with $t_k > 0$ and $\lim_{k\to\infty} t_k = 0$. Then $\{x(t_k) \mid k = 1, 2, \ldots\}$ is a convergent sequence and $\lim_{k\to\infty} x(t_k) = x^*$. Theorem 3 shows that $x(t_k)$ is an ε_k-perfect stationary point of f for all $k \geq k^*$ when k^* is sufficiently large. Thus, $\{x(t_k) \mid k \geq k^*\}$ is a sequence of ε_k-perfect stationary point of f on P converging to x^*. Therefore, we obtain from Definition 1 that x^* is a perfect stationary point of f on P. □

3 Numerical Results

3.1 Adaptation of a Predictor-Corrector Method for Numerically Following the Path

In this section, we adapt a predictor-corrector method for following the smooth path developed in the previous section. The predictor-corrector method has been applied to numerically trace the smooth path constructed by the homotopy methods that are referred to the literature such as Allgower and Georg (1990) and Eaves and Schmedders (1999).

We first parameterize (x, z, s, t) with the path length ξ, so that $x = x(\xi)$, $z = z(\xi)$, $s = s(\xi)$, and $t = t(\xi)$. Let $Dp_\alpha(x(\xi), z(\xi), s(\xi), t(\xi))$ denote the Jacobian matrix of $p_\alpha(x(\xi), z(\xi), s(\xi), t(\xi))$. Then, consider the initial-value problem given by

$$Dp_\alpha(x(\xi), z(\xi), s(\xi), t(\xi))(\tfrac{dx(\xi)}{d\xi}, \tfrac{dz(\xi)}{d\xi}, \tfrac{ds(\xi)}{d\xi}, \tfrac{dt(\xi)}{d\xi})^\top = 0,$$

$$\left\|(\tfrac{dx(\xi)}{d\xi}, \tfrac{dz(\xi)}{d\xi}, \tfrac{ds(\xi)}{d\xi}, \tfrac{dt(\xi)}{d\xi})\right\|_2 = 1,$$

$$\text{sign}(\det\begin{pmatrix} Dp_\alpha(x(\xi), z(\xi), s(\xi), t(\xi)) \\ (\frac{dx(\xi)}{d\xi}, \frac{dz(\xi)}{d\xi}, \frac{ds(\xi)}{d\xi}, \frac{dt(\xi)}{d\xi}) \end{pmatrix}) = b_0,$$

$$x(0) = x_0, z(0) = z_0, s(0) = s_0, t(0) = 1,$$

where $b^0 \in \{-1, 1\}$ is given by

$$b^0 = \text{sign}(\det\begin{pmatrix} Dp_\alpha(x_0, z_0, s_0, t_0) \\ (\frac{dx_0(\xi)}{d\xi}, \frac{dz_0(\xi)}{d\xi}, \frac{ds_0(\xi)}{d\xi}, \frac{dt_0(\xi)}{d\xi}) \end{pmatrix}),$$

with $(\frac{dx_0(\xi)}{d\xi}, \frac{dz_0(\xi)}{d\xi}, \frac{ds_0(\xi)}{d\xi}, \frac{dt_0(\xi)}{d\xi})$ being the solution of

$$Dp_\alpha(x_0, z_0, s_0, t_0)(\tfrac{dx(\xi)}{d\xi}, \tfrac{dz(\xi)}{d\xi}, \tfrac{ds(\xi)}{d\xi}, \tfrac{dt(\xi)}{d\xi})^\top = 0,$$

$$\left\|(\tfrac{dx(\xi)}{d\xi}, \tfrac{dz(\xi)}{d\xi}, \tfrac{ds(\xi)}{d\xi}, \tfrac{dt(\xi)}{d\xi})\right\|_2 = 1,$$

$$\tfrac{dt(\xi)}{d\xi} < 0.$$

Let $Dp_\alpha(\xi)$ stand for $Dp_\alpha(x(\xi), z(\xi), s(\xi), t(\xi))$ and let $g(Dp_\alpha(\xi))$ be the solution of

$$Dp_\alpha(\xi)g = 0,$$

$$\|g\|_2 = 1,$$

$$\text{sign}(\det\begin{pmatrix} Dp_\alpha(\xi) \\ g^\top \end{pmatrix}) = b_0.$$

Then

$$g(Dp_\alpha(\xi)) = \frac{(I - Dp_\alpha(\xi)^\top(Dp_\alpha(\xi)Dp_\alpha(\xi)^\top)^{-1}Dp_\alpha(\xi))p}{\|(I - Dp_\alpha(\xi)^\top(Dp_\alpha(\xi)Dp_\alpha(\xi)^\top)^{-1}Dp_\alpha(\xi))p\|_2},$$

where I is the $(2m+n+1)\times(2m+n+1)$ identity matrix and p is any vector of R^{2m+n+1} satisfying

$$\text{sign}(\det\begin{pmatrix} Dp_\alpha(\xi) \\ g(Dp_\alpha(\xi))^\top \end{pmatrix}) = b_0.$$

To solve the above initial value problem, we adopt a predictor-corrector method, which is as follows.

Intialization: Let $t_0 = 1$. Compute $Dp_\alpha^0 = Dp_\alpha(x_0, z_0, s_0, t_0)$. Choose an arbitrary vector $p \in R^{2m+n+1}$ satisfying

$$((I - Dp_\alpha^{0\top}(Dp_\alpha^0 Dp_\alpha^{0\top})^{-1}Dp_\alpha^0)p) \neq 0.$$

The initial tangent vector is given by

$$g^0 = \frac{(I - Dp_\alpha^{0\top}(Dp_\alpha^0 Dp_\alpha^{0\top})^{-1}Dp_\alpha^0)p}{\|(I - Dp_\alpha^{0\top}(Dp_\alpha^0 Dp_\alpha^{0\top})^{-1}Dp_\alpha^0)p\|_2}.$$

If $g^0_{2m+n+1} > 0$, let $g^0 = -g^0$. Compute

$$b_0 = \text{sign}(\det\begin{pmatrix} Dp_\alpha^0 \\ g^{0\top} \end{pmatrix}).$$

Let ϵ be any given tolerance and δ a sufficiently small positive number. Let $k = 0$ and go to step 1.

Step 1: Choose a predictor-step length $\overline{d} > 0$ satisfying that $\left\| p_\alpha(\overline{x}, \overline{z}, \overline{s}, \overline{t}) \right\|_2 < \delta, \bar{z} > 0, \bar{s} > 0$, and $0 < \bar{t} < 1$, where

$$(\overline{x}, \overline{z}, \overline{s}, \overline{t})^\top = (x_k, z_k, s_k, t_k)^\top + \overline{d} g^k.$$

Solve the following system of nonlinear equations,

$$p_\alpha(x, z, s, t) = 0,$$

$$(x, z, s, t) g^k = (\overline{x}, \overline{z}, \overline{s}, \overline{t}) g^k,$$

using Newton's Method with $(\overline{x}, \overline{z}, \overline{s}, \overline{t})$ being the starting point. Let $(x_{k+1}, z_{k+1}, s_{k+1}, t_{k+1})$ be an approximate solution of the system and $k = k+1$. Go to Step 2.

Step 2: If $t_k < \epsilon$, then the method terminates and an approximate solution has been found. Otherwise, proceed as follows. Compute $Dp_\alpha(x_k, z_k, s_k, t_k)$. Let $Dp_\alpha^{(k)}$ stand for $Dp_\alpha(x_k, z_k, s_k, t_k)$. Choose an arbitrary vector p of R^{2m+n+1} satisfying that

$$((I - Dp_\alpha^{(k)\top}(Dp_\alpha^{(k)} Dp_\alpha^{(k)\top})^{-1} Dp_\alpha^{(k)})p) \neq 0.$$

Let

$$g^k = \frac{(I - Dp_\alpha^{(k)\top}(Dp_\alpha^{(k)} Dp_\alpha^{(k)\top})^{-1} Dp_\alpha^{(k)})p}{\left\| (I - Dp_\alpha^{(k)\top}(Dp_\alpha^{(k)} Dp_\alpha^{(k)\top})^{-1} Dp_\alpha^{(k)})p \right\|_2}.$$

Compute

$$b_k = \text{sign}(\det \begin{pmatrix} Dp_\alpha^{(k)} \\ g^{k\top} \end{pmatrix}).$$

If $b_k \neq b_0$, let $g^k = -g^k$. Go to Step 1.

The method is a standard predictor-corrector method. Its convergence analysis is referred to Allgower and Georg (1990). As mentioned in Section

2, the path leads to some point $(x^*, z^*, s^*, 0)$ on the target level $t = 0$ such that $p_\alpha(x^*, z^*, s^*, 0) = 0$.

3.2 Numerical Performance

The method is coded in MATLAB (7.6.0). The following examples illustrate how the smooth path leads to a perfect stationary point of a polynomial mapping on a polytope.

Example 2. Take $n = 2$ and $a_1 = -a_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a_2 = -a_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $b_1 = b_2 = 1$, and $b_3 = b_4 = 0$. Then $P = \{x \in R^2 \mid 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$. Let $f : P \mapsto R^2$ be given *by* $f(x) = \begin{pmatrix} x_1 \\ -1 \end{pmatrix}$.

This example is the same as the one in Section 2 to demonstrate the concept of perfect stationary point. It has two stationary points, which are $(0, 0)^\top$ and $(1, 0)^\top$. But only $(1, 0)^\top$ is a perfect stationary point.

The initial predictor step length was set to be $d = 0.1$ and $\eta = 0.25$. Figure 1 illustrates the computing process for Example 2. In the figure, t denotes the artificial variable and x_1 and x_2 denote the coordinates of the stationary points of f on $P(t)$ for varying $t \in (0, 1]$. In the two upper graphs of Figure 1, the path starts from the right with $t = 1$, moves to the left, and approaches the perfect stationary point with $x_1 = 1$ and $x_2 = 0$ as t goes to zero, respectively. The lower graph of Figure 1 shows that the method starts from the unique solution of the trivial system $p_\alpha(x, z, s, 1) = 0$ with $(x_1, x_2) = (0.5, 0.5)$. It then takes 87 iterations to approach the perfect stationary point $(x_1, x_2) = (1, 0)$.

Example 3. $P = \{x \in R^3 \mid -x_j \le 0,\ j = 1, 2, 3,\ x_1 + x_2 + x_3 \le 1\}$ and

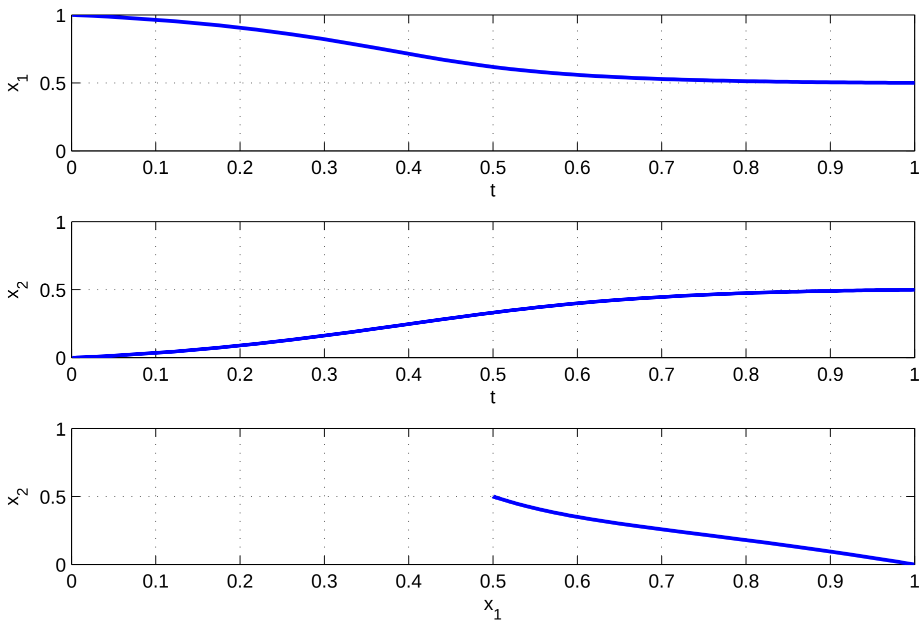

Figure 1: Illustration of Example 1

$f : P \to R^3$ is given by $f(x) = Mx$, where

$$M = \begin{pmatrix} 11 & 10 & 1 \\ 10 & 10 & 3 \\ 1 & 3 & 3 \end{pmatrix}.$$

This example is derived from Example 3 in van der Laan et al. (2006). We slightly modify the example by expanding its constraint from the unit simplex to a one-dimension higher polytope. There are three stationary points while $x \in S^2 = \{x \in R^3_+ \mid \sum_{i=1}^{3} x_i = 1\}$, namely all three unit vectors $e^1 = (1,0,0)^\top$, $e^2 = (0,1,0)^\top$, $e^3 = (0,0,1)^\top$. The first two are also perfect stationary points of f on P.

Figure 2 and Figure 3 illustrate the computing process for Example 3. Set $\eta = 0.125$ and initial step $d = 0.1$ in this example. In the graphs of

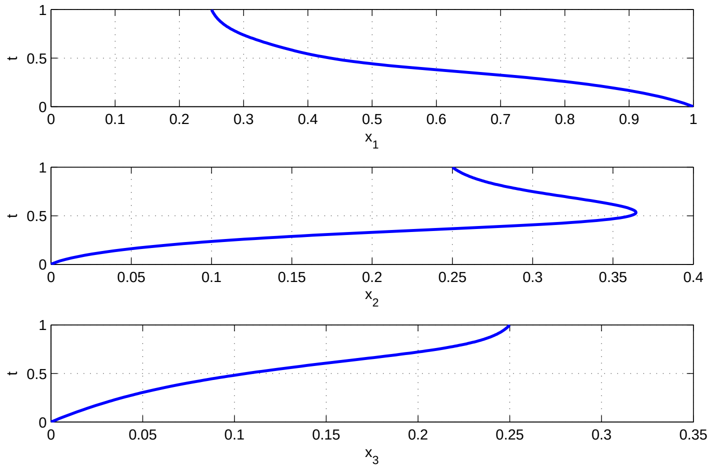

Figure 2: Illustration of Example 3

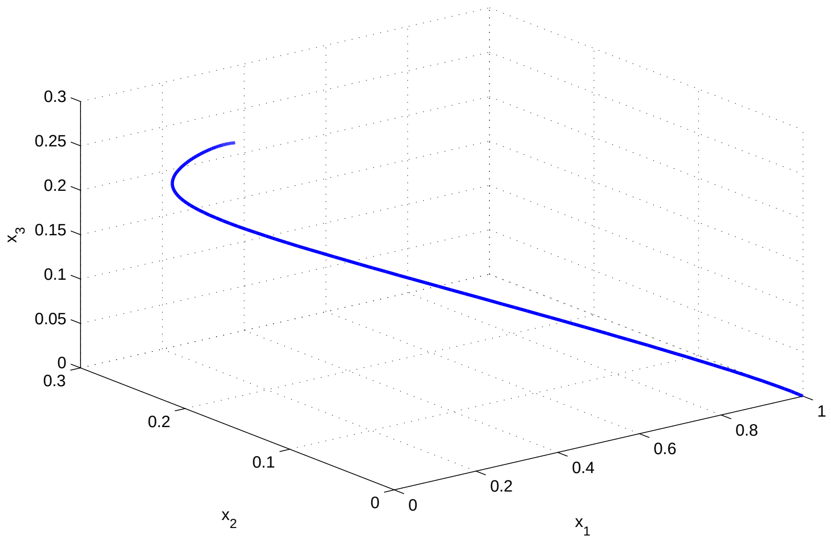

Figure 3: Stationary point on the smooth path of Example 3

Figure 2, the path starts from the top with $t = 1$ and approaches the bottom with $t = 0$ at the unique perfect stationary point where $x_1 = 1$, $x_2 = 0$, and

$x_3 = 0$, respectively. In Figure 3, the method starts from the unique solution of the trivial system $p_\alpha(x, z, s, 1) = 0$ with $(x_1, x_2, x_3) = (0.25, 0.25, 0.25)$ and it takes 187 iterations to approach the perfect stationary point $(1, 0, 0)$.

Example 4. [1] $P = \{x \in R^2 \mid 1 - x_1 \leq 0, \ - x_2 \leq 0, \ x_1 + x_2 \leq 2\}$ and $f : P \to R^2$ is given by $f(x) = ((x_1 + 1)^2, 1)^\top$.

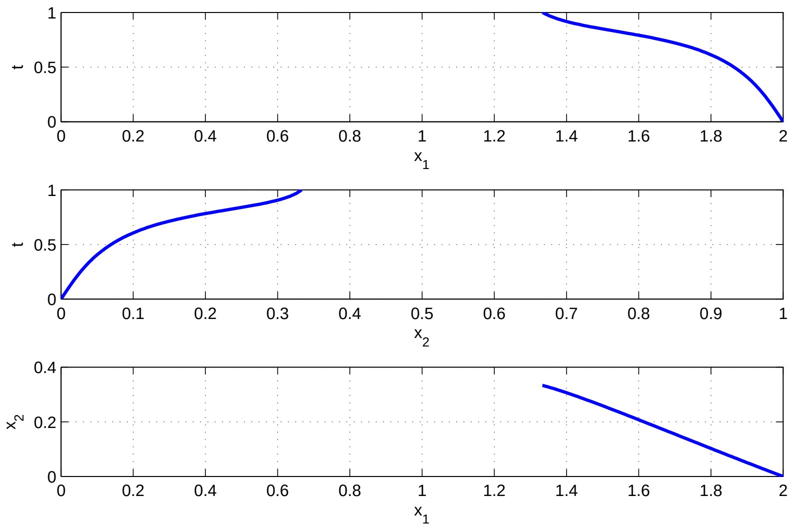

Figure 4: Illustration of Example 4

The initial predictor step length was set to be $d = 0.1$ and $\eta = 0.1$. Figure 4 illustrates the computing process for Example 4. In the two upper graphs of Figure 4, the path starts from the top with $t = 1$ and approaches the bottom with $t = 0$ at the point x where $x_1 = 2$ and $x_2 = 0$, respectively. In the lower graph of Figure 4, the method starts from the unique solution of the trivial system $p_\alpha(x, z, s, 1) = 0$ with $(x_1, x_2) = (1.3334, 0.3333)$ and it

[1] This example derives from the Example 4.1 in Fan and Yu(2009). We modify the example by adding a boundary constraint for the variables.

takes 112 iterations in total to approach the point $(2,0)$. To show $(2,0)$ is a perfect stationary point, for $\varepsilon \in (0,1)$, let

$$x_1(\varepsilon) = 1 + (1-\varepsilon)^2,\ x_2(\varepsilon) = \varepsilon.$$

Clearly, $x(\varepsilon)$ lies in the interior of P for each $\varepsilon \in (0,1)$. Then $f(x(\varepsilon)) = z_2 a_2 + z_3 a_3 = z_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + z_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with $z_2 = (2 + (1-\varepsilon)^2)^2 - 1 > 0$ and $z_3 = (2 + (1-\varepsilon)^2)^2 > 0$, and it holds that $(0,-1) \begin{pmatrix} 1 + (1-\varepsilon)^2 \\ \varepsilon \end{pmatrix} = -\varepsilon \geq 0 - \varepsilon$ and $(1,1) \begin{pmatrix} 1 + (1-\varepsilon)^2 \\ \varepsilon \end{pmatrix} = 2 - \varepsilon + \varepsilon^2 \geq 2 - \varepsilon$. Hence, for each $\varepsilon \in (0,1)$, $x(\varepsilon)$ is an ε-perfect stationary point of f. It is easy to see that $\lim\limits_{\varepsilon \to 0} x(\varepsilon) = (2,0)$.

Example 5. [2] $P = \{x \in R^2 \mid x_1 \geq -1,\ x_2 \leq 0,\ x_1 \leq x_2\}$ and $f : P \to R^2$ is given by $f(x) = ((2x_1 - 10)^2, 2x_2)^\top$.

The initial predictor step length was set to be $d = 2$ and $\eta = 0.1$. Figure 5 illustrates the computing process for Example 5. From the figure one can see that the value of the two variables varies rather slowly with the change of parameter t, therefore we set the initial predictor step length quite large in the beginning. In the two upper graphs of Figure 5, the path starts from the top with $t = 1$ and approaches the bottom with $t = 0$ at the perfect point x where $x_1 = 0$ and $x_2 = 0$, respectively. The lower graph of Figure 5 shows that the method starts from the unique solution of the trivial system $p_\alpha(x, z, s, 1) = 0$ with $(x_1, x_2) = (-2/3, -1/3)$ and it takes 126 iterations in total to approach the point $(0,0)$. To show $(0,0)$ is a perfect stationary point, for $\varepsilon \in (0,1)$, let

$$x_1(\varepsilon) = -\varepsilon,\ x_2(\varepsilon) = -\varepsilon.$$

[2]This example is derived from Example 4.2 in Fan and Yu (2009). We modify the example by adding two boundary constraints for the variables.

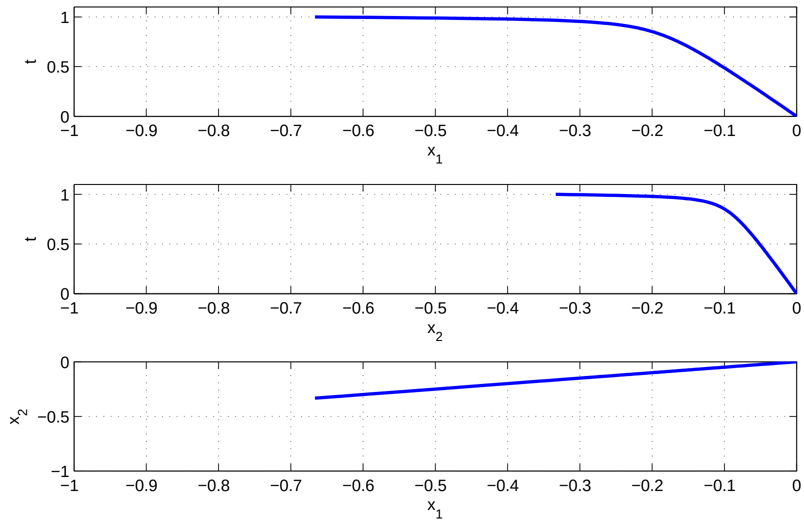

Figure 5: Illustration of Example 5

Clearly, $x(\varepsilon)$ lies in the interior of P for each $\varepsilon \in (0,1)$. Then $f(x(\varepsilon)) = z_2a_2 + z_3a_3 = z_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + z_3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with $z_2 = (2\varepsilon+10)^2 - 2\varepsilon > 0$ and $z_3 = (2\varepsilon+10)^2 > 0$. It also holds that $(0,1)\begin{pmatrix} -\varepsilon \\ -\varepsilon \end{pmatrix} = -\varepsilon \geq 0 - \varepsilon$ and $(1,-1)\begin{pmatrix} -\varepsilon \\ \varepsilon \end{pmatrix} = 0 \geq -\varepsilon$. Hence, for each $\varepsilon \in (0,1)$, $x(\varepsilon)$ is an ε-perfect stationary point of f. It is easy to see that $\lim_{\varepsilon\to 0} x(\varepsilon) = (0,0)$.

Example 6. [3] $P = \{x \in R^4 | x_i \geq 0, i = 1,2,3,4, x_1+x_2+x_3+x_4 \leq 3\}$ and $f : P \to R^4$ is given by $f(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_2^2 + x_1 + 3x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 3x_4 - 1 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}$.

The initial predictor step length was set to be $d = 0.1$ and $\eta = 0.3$. Figure 6 illustrates the computing process for Example 6. In Figure 6, the path of

[3] This example is derived from the Kojima-Shindo nonlinear complementarity test problem. We modify the example by adding upper bound constraints for all variables.

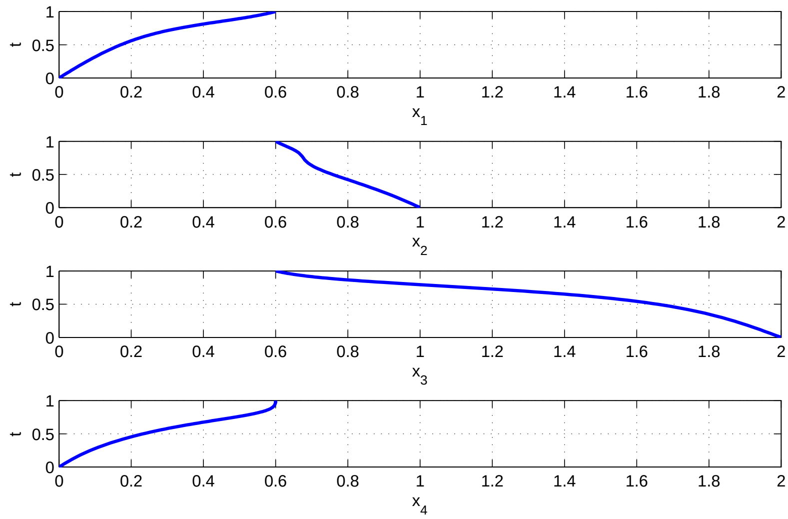

Figure 6: Illustration of example 6

the four variables starts from the top with $t = 1$ and approaches the bottom with $t = 0$ at the perfect stationary point x where $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, and $x_4 = 0$ after 126 iterations in total.

Example 7. $P = \{x \in R^n |\ x_i \geq 1, i = 1, 2, ..., n,\ \sum_{i=1}^{n} x_i \leq 2n\}$ and $f : P \to R^n$ is given by $f(x) = Mx + r$, where $M = UDU^\top$ is an $n \times n$ matrix, with D being a diagonal matrix with diagonal elements selected randomly from a uniform distribution on $(0, 1)$ and $U = I_n - \frac{2}{\|z\|^2} zz^\top$ with each vector $z \in R^n$ randomly generated.

Table 1 shows the average time in seconds and average number of iterations required on a 2.00 GHz Windows PC. For each setting of (m, n, q), 10 examples were randomly generated and solved. From these numerical results, one can see that the method seems efficient.

Table 1: Average no. of iterations and computational time

n	m	q	initial step	avg. no. of iterations	avg. comp. time
5	6	2	0.5	34.6	2.0058
10	11	2	0.5	60	5.8948
20	21	2	0.5	90.4	16.4391
30	31	2	0.5	201.6	53.4689
50	51	2	0.5	176.3	49.1039
100	101	2	0.5	483.5	1108.7946

4 Conclusion

This paper studies the stationary point problem from the perspective of stability. We fully exploit the differentiability of a stationary point problem and develop an interior-point path-following method for computing a perfect stationary point of a polynomial mapping on a polytope. We construct a smooth path which leads to a perfect point of a polynomial mapping on a polytope. A predictor-corrector method is adopted for numerically following the path and numerical results confirm the effectiveness of the method. How to compute a proper stationary point of a polynomial mapping on a polytope by the interior-point path-following method will be explored in future work.

Appendix

This appendix proves that the Jacobian matrix $Dp(x, z, s, t; \alpha)$ of $p(x, z, s, t; \alpha)$ if of full-row rank for any $(x, z, s, t; \alpha) \in int(P(t)) \times R^m_{++} \times R^m_{++} \times (0, 1] \times R^n$. This property is used in the proof of Theorem 2.

We compute the Jacobian matrix $Dp(x, z, s, t; \alpha)$ of $p(x, z, s, t; \alpha)$ with respect to $\omega = (x, z, s, t; \alpha) \in int(P(t)) \times R^m_{++} \times R^m_{++} \times (0, 1] \times R^n$.

When $t = 1$,

$$p(x,z,s,t;\alpha) = \begin{cases} -\sum_{i=1}^{m} z_i a_i, \\ \\ a_j^\top + s_j + \eta - b_j,\ j = 1,2,...,m, \\ \\ z_j s_j - 1,\ j = 1,2,...,m. \end{cases}$$

So,

$$J = \frac{\partial p}{\partial \omega} = \begin{pmatrix} 0 & -A^\top & 0 & 0 & 0 \\ A & 0 & I_m & 0 & 0 \\ 0 & S & Z & 0 & 0 \end{pmatrix} = (J_1, \mathbf{0}),$$

where I_m is the $m \times m$ identity matrix, S and Z are $m \times m$ diagonal matrices, with their i-th elements given by s_i and z_i, respectively. Note that $s_i > 0$ and $z_i > 0$ for all i because of $s_j z_j = 1$, so S and Z are invertible. Right-multiplying J_1 by

$$\Lambda_1 = \begin{pmatrix} I_n & 0 & 0 \\ S^{-1}ZA & I_m & 0 \\ -A & -Z^{-1}S & I_m \end{pmatrix}$$

we obtain

$$J_1\Lambda_1 = \begin{pmatrix} -A^\top S^{-1}ZA & -A^\top & 0 \\ 0 & -Z^{-1}S & I_m \\ 0 & 0 & Z \end{pmatrix}.$$

Note that $A^\top S^{-1}ZA$ is positive definite and $J_1\Lambda_1$ is an upper triangular matrix. So $J_1\Lambda_1$ is non-singular. Since Λ_1 is, as it is a lower triangular matrix with identity matrices on its diagonal, also non-singular, we have that $J_1 = (J_1\Lambda_1)(\Lambda_1^{-1})$ is non-singular. Thus, $Dp(x,z,s,1;\alpha) = J$ is of full-row rank.

When $t \in (0,1)$, $p(x,z,s,t;\alpha)$ is the left side of system (3). The Jacobian matrix $J = Dp(x,z,s,t;\alpha)$ of $p(x,z,s,t;\alpha)$ is given by

$$J = \begin{pmatrix} (1-t^q)\partial_x f(\cdot) & -A^\top & 0 & c_t & -t^n(1-t^n)I_n \\ A & 0 & I_m & \eta e & 0 \\ 0 & S & Z & -qt^{q-1}e & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, $e = (1, 1, ..., 1)^\top$ is the n-vector of ones, and $c_t(i) = -(f_i(x) + \alpha_i)qt^{q-1} + \alpha_i 2qt^{2q-1}$ with $i = 1, 2, ..., n$. Applying row and column operation, one can easily reduce $Dp(x, z, s, t; \alpha)$ to

$$J_2 = \begin{pmatrix} (1-t^q)\partial_x f(\cdot) & c_t & -A^\top & 0 & -t^n(1-t^n)I_n \\ A & \eta e & 0 & I_m & 0 \\ -ZA & -qt^{q-1}e - \eta Ze & S & 0 & 0 \end{pmatrix}.$$

Thus, for any $t \in (0, 1), Dp(x, z, s, t; \alpha)$ is of full-row rank. This completes the proof.

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